

# Corner Transfer Matrix Renormalization Group Method Applied to the Ising Model on the Hyperbolic Plane

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(Received )

Critical behavior of the Ising model is investigated at the center of large scale finite size systems, where the lattice is represented as the tiling of pentagons. The system is on the hyperbolic plane, and the recursive structure of the lattice makes it possible to apply the corner transfer matrix renormalization group method. From the calculated nearest neighbor spin correlation function and the spontaneous magnetization, it is concluded that the phase transition of this model is mean-field like. One parameter deformation of the corner Hamiltonian on the hyperbolic plane is discussed.

KEYWORDS: CTMRG, Ising Model, Scaling, Curvature

## §1. Introduction

Baxter's method of corner transfer matrix (CTM) has been known as one of the representative tool for analytical study of statistical models in two dimension (2D).<sup>1-3)</sup> The method is also of use for numerical calculations of one point functions, such as the local energy and the magnetization.<sup>3)</sup> This numerical application is a kind of numerical renormalization group (RG) method, where the block spin transformation is obtained from the diagonalization of CTMs. Such a RG scheme has many aspects in common with the density matrix renormalization group (DMRG) method,<sup>4-7)</sup> especially when the method is applied to 2D classical lattice models.<sup>8)</sup>

Introducing the flexibility in the system extension process of the DMRG method to the Baxter's method of CTM, the authors developed the corner transfer matrix renormalization group (CTMRG) method.<sup>9-12)</sup> In this article we report a modification of the CTMRG method, for the purpose of applying the method to classical lattice models on the hyperbolic plane. Using the recursive structure of the lattice, we obtain one point functions at the center of sufficiently large finite size systems.

Quite recently Hasegawa, Sakaniwa, and Shima reported deviations of critical indices of the Ising model on the hyperbolic plane from the well known Ising universality classes in two dimension.<sup>14,15)</sup> They predicted that phase transition of such systems would be mean-field like. To confirm their prediction, in the next section we consider the Ising model on a lattice, which is represented as the tiling of pentagons. The necessary modification of the CTMRG method on this lattice is explained in §3. We calculate the nearest neighbor spin correlation function and the spontaneous magnetization at the center of large scale finite size systems. Critical indices for these one point functions are studied in §4, and we confirm the mean-field like properties of the phase transition. Conclusions are summarized in the last section. We discuss a possible deformation of corner Hamiltonian in the hy-

perbolic plane.

## §2. Ising Model on the Tiling of Pentagons

Let us consider the hyperbolic plane, which is the two dimensional surface with constant negative curvature. Figure 1 shows a part of a sufficiently large regular lattice on the plane, where the lattice is constructed as the tiling of pentagons.<sup>13)</sup> All the arcs are geodesics, which divides the lattice into two parts of similar structure. Each lattice point represented by an open circle is the crossing points of two geodesics.

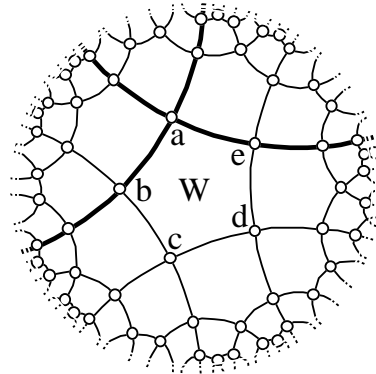


Fig. 1. Ising model on a regular lattice on the hyperbolic plane. Open circles represent Ising spins on the lattice point, and  $W(\sigma_a, \sigma_b, \sigma_c, \sigma_d, \sigma_e)$  is the local Boltzmann weight defined in Eq. (2.2). Two geodesics drawn by thick arcs divide the system into four quadrants, which are called as the corner.

We investigate the ferromagnetic Ising model on this lattice. The Hamiltonian of the system is defined as the sum of nearest neighbor Ising interactions

$$H = -J \sum_{\langle i, j \rangle} \sigma_i \sigma_j, \quad (2.1)$$

where  $\langle i, j \rangle$  represents pair of neighboring sites, and  $\sigma_i = \pm 1$  and  $\sigma_j = \pm 1$  are the Ising spins on the lattice

points. Throughout this article we assume the absence of external magnetic field. For the latter conveniences, we represent the system as the ‘interaction round a face (IRF)’ model. The local Boltzmann weight for each face of pentagonal shape — the IRF weight — is given by

$$W(\sigma_a, \sigma_b, \sigma_c, \sigma_d, \sigma_e) = \exp \left\{ -\frac{\beta J}{2} (\sigma_a \sigma_b + \sigma_b \sigma_c + \sigma_c \sigma_d + \sigma_d \sigma_e + \sigma_e \sigma_a) \right\} \quad (2.2)$$

where  $\beta = 1/k_B T$  is the inverse temperature, and  $\sigma_a, \sigma_b, \sigma_c, \sigma_d$ , and  $\sigma_e$  are the spin variables around the face as shown in Fig. 1.

The partition function of a finite size system (with sufficiently large diameter) is formally written as the configuration sum of the Boltzmann weight of the whole system

$$Z = \sum_{\text{all the spins}} \prod_{\text{all the faces}} W, \quad (2.3)$$

where we are interested in the thermodynamic limit of this system. Note that it is rather hard to investigate the system by use of the Monte Carlo simulations, since the number of sites contained in a cluster blows up exponentially with respect to its diameter. As a complementary numerical tool, we employ the CTMRG method.

### §3. Corner Transfer Matrix Renormalization Group Method

The two geodesics shown by thick arcs in Fig. 1 divide the system into four parts, which are called as corners.<sup>3)</sup> Figure 2 shows the structure of a corner. We label the spins on a cut as  $\{\sigma_1, \sigma_2, \sigma_3, \dots\}$ , and those on another cut as  $\{\sigma'_1, \sigma'_2, \sigma'_3, \dots\}$ , where  $\sigma_1$  is equivalent to  $\sigma'_1$ . The corner transfer matrix is the Boltzmann weight with respect to a corner, which is calculated as a partial sum of the product of IRF weights in the corner

$$C(\sigma'_1, \sigma'_2, \sigma'_3, \dots | \sigma_1, \sigma_2, \sigma_3, \dots) = \sum_{\substack{\text{spins inside} \\ \text{the quadrant}}} \prod_{\substack{\text{faces in} \\ \text{the quadrant}}} W. \quad (3.1)$$

The configuration sum is taken over spins ‘inside’ the corner, leaving those spins on the cuts. Conventionally the matrix  $C$  is interpreted as block diagonal with respect to  $\sigma_1$  and  $\sigma'_1$ , and the element  $C(\sigma'_1, \sigma'_2, \sigma'_3, \dots | \sigma_1, \sigma_2, \sigma_3, \dots)$  for those cases  $\sigma_1 \neq \sigma'_1$  is set to zero. The CTM thus defined is symmetric in the case of the pentagonal lattice under consideration.

As shown in Fig. 2 a corner has the structure where three parts labeled by  $\bar{P}$  and two parts labeled by  $\bar{C}$  are joined to a face  $W$ . The ‘fusion’ relation can be represented by a formal equation<sup>9, 10)</sup>

$$C = W \cdot \bar{P} \bar{C} \bar{P} \bar{C}. \quad (3.2)$$

Note that  $\bar{C}$  is a corner of smaller size.

For convenience, let us observe the structure of the part of the system shown in Fig. 3, where two  $\bar{P}$ , and  $\bar{C}$  are joined to  $W$ . Labeling the shown part by  $P$ , we can

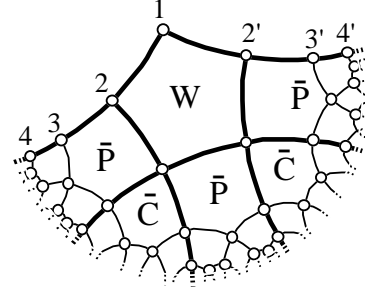


Fig. 2. Recursive structure of a corner  $C$ .

formally write the fusion relation in the same manner

$$P = W \cdot \bar{P} \bar{C} \bar{P}. \quad (3.3)$$

For a conventional reason we call  $P$  as the ‘half-row’, although  $P$  is not a row on the hyperbolic plane. It is easily understood that  $\bar{P}$  is a half-row of smaller size. We have thus obtained recursive structure of the corner  $C$  and the half-row  $P$ .

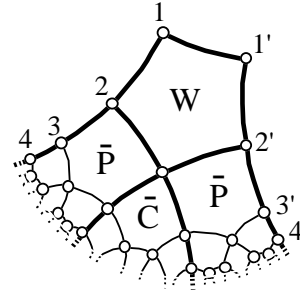


Fig. 3. Recursive structure of the ‘half-row’  $P$ .

As we have defined CTM for a corner, let us express the Boltzmann weight with respect to the half-row

$$P(\sigma'_1, \sigma'_2, \sigma'_3, \dots | \sigma_1, \sigma_2, \sigma_3, \dots) = \sum_{\substack{\text{spins inside} \\ \text{the half-row}}} \prod_{\substack{\text{faces in} \\ \text{the half-row}}} W \quad (3.4)$$

in the matrix form, where the positions of spins  $\{\sigma_1, \sigma_2, \sigma_3, \dots\}$  and  $\{\sigma'_1, \sigma'_2, \sigma'_3, \dots\}$  are shown in Fig. 3. We call the weight in the matrix form as the half-row transfer matrix (HRTM).

The 4-th power of the CTM

$$\rho = C^4 \quad (3.5)$$

is a kind of density matrix, since its trace gives the partition function

$$Z = \text{Tr } \rho = \text{Tr } C^4 \quad (3.6)$$

of a finite size cluster that consists of four corners. The

matrix dimension of CTM, and also that of the HRTM, increases exponentially with respect to the system size. In order to obtain  $Z$  numerically up to sufficiently large systems, we introduce the block spin transformation that is created from the diagonalization of the density matrix  $\rho$ .<sup>4,5)</sup>

Instead of directly diagonalizing  $\rho$  in Eq. (3.5), we first create its contraction

$$\begin{aligned} & \rho'(\sigma'_2, \sigma'_3 \dots | \sigma_2, \sigma_3, \dots) \\ &= \sum_{\sigma'_1 = \pm 1} \rho(\sigma'_1, \sigma'_2, \sigma'_3 \dots | \sigma_1, \sigma_2, \sigma_3, \dots) \end{aligned} \quad (3.7)$$

and then diagonalize it

$$\begin{aligned} & \rho'(\sigma'_2, \sigma'_3 \dots | \sigma_2, \sigma_3, \dots) \\ &= \sum_{\xi} A(\sigma'_2, \sigma'_3 \dots | \xi) \lambda_{\xi} A(\sigma_2, \sigma_3, \dots | \xi), \end{aligned} \quad (3.8)$$

where the eigenvalue  $\lambda_{\xi}$  is non-negative. Following the convention in DMRG, we assume the decreasing order for  $\lambda_{\xi}$ . The orthogonal matrix  $A(\sigma_2, \sigma_3, \dots | \xi)$  represents the block spin transformation from the ‘row-spin’  $\{\sigma_2, \sigma_3, \dots\}$  to the effective spin variable  $\xi$ . We keep  $m$  numbers of representative states, which correspond to major eigenvalues, for the block spin variable  $\xi$ . Applying the matrix  $A$  to the CTM and the HRTM, we obtain ‘renormalized matrices’ of  $2m$ -dimension

$$\begin{aligned} C(\sigma'_1, \sigma'_2, \dots | \sigma_1, \sigma_2, \dots) &\rightarrow C(\sigma', \xi' | \sigma, \xi) \\ P(\sigma'_1, \sigma'_2, \dots | \sigma_1, \sigma_2, \dots) &\rightarrow P(\sigma', \xi' | \sigma, \xi), \end{aligned} \quad (3.9)$$

where we have dropped the indices from  $\sigma_1$  and  $\sigma'_1$ .<sup>9,10)</sup>

Combining the recursive structures in Eqs. (3.2) and (3.3), and the renormalization scheme in Eqs. (3.7)-(3.9), we can obtain the CTM and HRTM in the renormalized form for arbitrary system size by way of successive extension of the system.

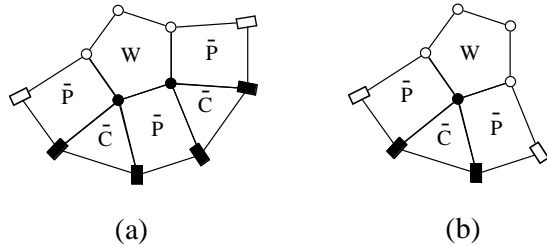


Fig. 4. Extension process of (a) CTM and (b) HRTM in the renormalized expression.

Suppose that we have  $C(\sigma', \xi' | \sigma, \xi)$  and  $P(\sigma', \xi' | \sigma, \xi)$  for a finite size cluster. In order to explain the extension process, let us rewrite these matrices as  $\bar{C}(s', \zeta' | s, \zeta)$  and  $\bar{P}(s', \zeta' | s, \zeta)$ .

- (1) Substitute  $\bar{C}(s', \zeta' | s, \zeta)$  and  $\bar{P}(s', \zeta' | s, \zeta)$  into the fusion process in Eqs. (3.2) and (3.3). Figure 4 shows these fusion processes among  $W$ ,  $\bar{C}$ , and  $\bar{P}$ , where rectangles correspond to the block spin vari-

ables. The spin variables that are contracted out are shown by black marks. As a result, we obtain the extended CTM  $C(\sigma', s', \zeta' | \sigma, s, \zeta)$  and the extended HRTM  $P(\sigma', s', \zeta' | \sigma, s, \zeta)$ .

- (2) From the extended CTM  $C(\sigma', s', \zeta' | \sigma, s, \zeta)$  obtain the density matrix  $\rho(\sigma', s', \zeta' | \sigma, s, \zeta)$  by Eq. (3.5). Contracting out the spin at the center, obtain  $\rho'(s', \zeta' | s, \zeta)$  as Eq. (3.7), and diagonalizing it to obtain the block spin transformation matrix  $A(s, \zeta | \xi)$  from Eq. (3.8).
- (3) Applying  $A(s, \zeta | \xi)$  to both  $C(\sigma', s', \zeta' | \sigma, s, \zeta)$  and  $P(\sigma', s', \zeta' | \sigma, s, \zeta)$ , obtain the extended CTM  $C(\sigma', \xi' | \sigma, \xi)$  in the initial form, and the same for HRTM to obtain  $P(\sigma', \xi' | \sigma, \xi)$ .
- (4) return to the first step.

The system size, which is the length of the longest geodesics in the system, increases by 2 for each iteration.<sup>16)</sup> In order to start the above extension process, we set the initial condition

$$\bar{C}(\sigma' | \sigma) = \bar{P}(\sigma' | \sigma) = \delta(\sigma' | 1) \delta(\sigma | 1) \quad (3.10)$$

that represents ferromagnetic boundary, where  $\delta(a|b) = \delta_{a,b}$  is the Kronecker's delta.

During the iteration we can obtain one point functions at the center of the system. For example, the spontaneous magnetization is calculated as

$$\langle \sigma \rangle = \frac{\text{Tr } \sigma \rho}{\text{Tr } \rho} = \frac{\sum_{\sigma, s, \zeta} \sigma \rho(\sigma, s, \zeta | \sigma, s, \zeta)}{\sum_{\sigma, s, \zeta} \rho(\sigma, s, \zeta | \sigma, s, \zeta)}. \quad (3.11)$$

In the same manner we obtain the nearest neighbor spin correlation function  $\langle \sigma s \rangle$ . It should be noted that one point functions thus calculated at the center do not always represent the averaged property of the whole system even in the thermodynamic limit, since the area near the boundary has non-negligible weight in the hyperbolic plane.

#### §4. Numerical Result Compared With the Bethe Approximation

Let us calculate the spontaneous magnetization  $\langle \sigma \rangle$ , and spin correlation function  $\langle \sigma s \rangle$  for the nearest spin pair. We regard the Ising interaction strength  $J$  as the energy unit, and use the temperature where the Boltzmann constant  $k_B$  is equal to unity. Most of the numerical calculations are performed keeping  $m = 40$  states. The dumping of the density matrix eigenvalues is very fast, and actually the calculated results with  $m = 10$  do not differ from those obtained with  $m = 40$  even at the critical temperature  $T_C$ . The iteration number required for the numerical convergence is at most 400000 for the calculated data points.

Figure 5 shows the calculated results. The square of the spontaneous magnetization  $\langle \sigma \rangle^2$  is a linear function of temperature in the neighborhood of  $T_C$ . From the behavior we estimate the transition temperature  $T_C = 2.799$ . The nearest neighbor spin correlation function

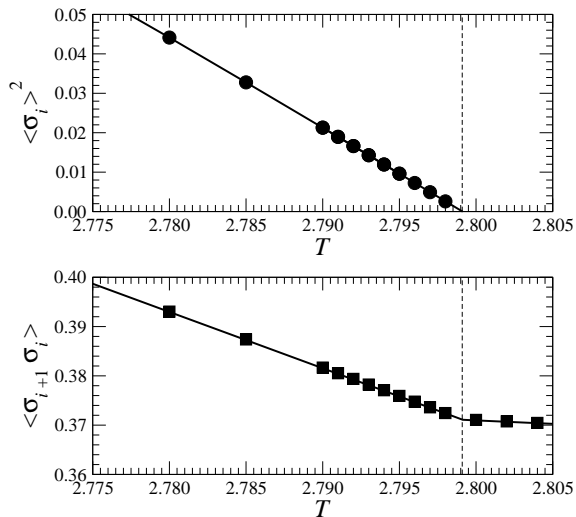


Fig. 5. Square of the spontaneous magnetization  $\langle \sigma \rangle^2$  (upper) and the nearest neighbor spin correlation function  $\langle \sigma s \rangle$  (lower) with respect to the temperature  $T$ .

$\langle \sigma s \rangle$  has a kink at  $T_C$ , and is linear in  $T$  around there. These calculated results support the existence of mean-field like transition, that is subject to the critical indices  $\beta = 1/2$  and  $\alpha = 0$ . We thus confirmed the prediction by Hasegawa, Sakaniwa, and Shima.<sup>14, 15)</sup>

Compared with the transition temperature of the square lattice Ising model  $T_C^{\text{Square}} = 2.269$ , the calculated  $T_C$  is fairly higher and is close to the transition temperature calculated from the Bethe approximation  $T_C^{\text{Bethe}} = 2.885$ .<sup>17, 18)</sup> The result suggest that neglect of the ‘loop back effect’ is not so conspicuous in the hyperbolic plane.

## §5. Conclusion and discussion

We have calculated the spontaneous magnetization and the nearest neighbor spin correlation function of the Ising model on a pentagonal lattice on the hyperbolic plane. The numerical algorithm of the CTMRG method is modified for this purpose. The calculated critical temperature is  $T_C = 2.799$ , and we observe the mean-field like phase transition.

The modified CTMRG method we have developed is applicable to regular lattices that consists of geodesics on the hyperbolic plane. For those lattices that does not contain geodesics, one has to either treating asymmetric density matrix or to draw geodesics by use of transformation such as duality transformation and the star-triangle relation. Generalization of the modified CTMRG method to the vertex model is straight forward. It may be interesting to classify ordered states of eight-vertex model on a variety of regular lattices in the hyperbolic plane.

An interest is in the eigenvalue structure of the density matrix at the transition temperature. Its analytic form is not well defined in the thermodynamic limit of classical lattice models on the flat 2D plane.<sup>19, 20)</sup> The rapid eigenvalue dumping observed on the hyperbolic plane suggests that there would be a way of regularizing the CTM at the criticality. The classical-quantum

correspondence from such a view point is worth considering. Formally speaking the corner transfer matrix  $C$  can be written as the exponential of the corner Hamiltonian  $H_C$ . When the lattice is on the flat plane, the simplest example of the corner Hamiltonian is written in the sum of local operators

$$\begin{aligned} H_C &= h(\sigma_1, \sigma_2) + 2h(\sigma_2, \sigma_3) + 3h(\sigma_3, \sigma_4) + \dots \\ &= h_1 + 2h_2 + 3h_3 + 4h_4 + \dots, \end{aligned} \quad (5.1)$$

where  $h_i = h(\sigma_i, \sigma_{i+1})$  is the local hamiltonian that acts between neighboring sites. A possible one parameter deformation of the corner Hamiltonian to the hyperbolic geometry may given by

$$H_C(\Lambda) = h_1 + \frac{\sinh 2\Lambda}{\sinh \Lambda} h_2 + \frac{\sinh 3\Lambda}{\sinh \Lambda} h_3 + \dots, \quad (5.2)$$

which is reduced to  $H_C$  in Eq. (5.1) in the limit  $\Lambda \rightarrow 0$ . The deformed Hamiltonian satisfies the recursive structure

$$\begin{aligned} H_C(\Lambda) &= \cosh \Lambda \left( h_2 + \frac{\sinh 2\Lambda}{\sinh \Lambda} h_3 + \frac{\sinh 3\Lambda}{\sinh \Lambda} h_4 + \dots \right) \\ &+ h_1 + \cosh \Lambda h_2 + \cosh 2\Lambda h_3 + \dots \end{aligned} \quad (5.3)$$

discussed in Okunishi’s RG scheme on the corner Hamiltonian.<sup>21)</sup> Such a deformation has similar regularization effect proposed by Okunishi quite recently.<sup>22)</sup>

T. N. and A. G is partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture. T. N. thank to Okunishi for valuable discussions.

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